

Von Neumann's consistency proof

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In 1925, the twenty-one-year-old John von Neumann wrote and submitted a long paper, 'Zur Hilbertschen Beweistheorie' (the paper would have been published two years later as von Neumann 1927), containing a consistency proof for a fragment of first-order arithmetic (the fragment without induction and with the successor axioms only) by a variant of Hilbert's substitution method. A correct estimate of the bounds on the substituents needed was given, and in fact it is built into the proof (while in most uses of such methods it is computed independently after the proof). The work also contained a clear presentation of Hilbert's proof-theoretic approach and a detailed critique of the most important consistency proof produced until then in the Hilbert School, namely Ackermann's proof. Moreover, von Neumann gave a very precise and thorough (though slightly peculiar) definition of formal system in general (in which, by the way, we find perhaps the first use in the literature of axiom schemas in the definition of a formal system), a rigorous delimitation of the specific formal system treated, together with many interesting side remarks (e.g., on decidability, on choice principles, on definitions by recursion, etc.). Our purpose is to describe the specific features of von Neumann's consistency proof, explaining some of its technical subtleties (the proof is not easy, as it is the case with substitution methods in general, but it needs no 'backtracking', a typical feature of such methods), trying to provide (at least in part) some missing explanations and heuristics (especially about the bounds imposed by von Neumann on the substituents needed in the induction step), and showing its originality with respect to the other proofs in the Hilbert School. We chose to concentrate strictly on the proof, and we did not even attempt to do justice to all the other themes and aspects of this very rich work of von Neumann's. Our choice is based on the fact that, to our knowledge, there is no detailed account whatever of this proof in the literature. There are, to be sure, a lot of scattered references to von Neumann's paper and to his result; one is even (in a footnote) on the first page of Goedel's incompleteness paper. Nevertheless, on the basis of the literature, one would conclude that after Ackermann (who briefly reviewed the work), and the few other experts at the time, nobody has looked at this proof in depth. We would like to fill this gap. Since the system whose consistency is proved by von Neumann is very weak (and its consistency can be proved much more easily with the methods of Herbrand or Gentzen, since one then has their general strong results at disposal), the main interest of his consistency proof lies in the method adopted by him, a method which is quite original in the context of the Hilbert School in the mid-Twenties, and whose possibilities of further develop-

ment and application (by adopting stronger background assumptions) have not been explored yet.

This is joint work with Lorenzo Carlucci

References

- [1] Von Neumann, J. (1927): 'Zur Hilbertschen Beweistheorie', *Mathematische Zeitschrift* 26, 1946