Reverse mathematics and well-scattered partial orders

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Abstract. I will discuss *well-scattered partial orders* in the context of reverse mathematics.

We say that P is a *well-scattered partial order* (WSPO) if for every function $f\colon \mathbb{Q} \to P$ there exist $x <_{\mathbb{Q}} y$ such that $f(x) \leq_P f(y)$. The following theorem gives four classically equivalent definitions for wspo's.

Theorem 1 (Bonnet, Pouzet 1969). *Let P be a partial order. Then the following are equivalent:*

1. *P is a wspo;*
2. *P is scattered and has no infinite antichains;*
3. *every linear extension of P is scattered;*
4. *for every function $f\colon \mathbb{Q} \to P$ there exists an infinite set $A \subseteq \mathbb{Q}$ such that $x <_{\mathbb{Q}} y$ implies $f(x) \leq_P f(y)$ for all $x, y \in A$.*

We want to compare the above definitions from the viewpoint of reverse mathematics. For example, what's necessary (and sufficient) to prove the statement "every linear extension of a wspo is scattered"?
It turns out that, except for the statements already provable in RCA_0, there are roughly two families: statements provable in WKL_0 but not in RCA_0 (neither in WWKL_0) and statements provable in ACA_0 but not in WKL_0. With regard to the second family, the following partition theorem comes into play.

Theorem 2 (Erdös, Rado 1952). *The partition relation $\mathbb{Q} \to (\aleph_0, \mathbb{Q})^2$ holds, that is for every coloring $c\colon [\mathbb{Q}]^2 \to 2$ there exists either an infinite 0-homogeneous set or a dense 1-homogeneous set.*

Let ER^2_2 (after Erdös-Rado) be the formal statement corresponding to Erdös-Rado theorem. The reverse mathematics of ER^2_2 is interesting by itself as it lies between ACA_0 and RT^2_2. I still do not know whether it is strictly between them. As for the relation with wspo's, I will discuss semitransitive versions of ER^2_2.

Keywords: Reverse Mathematics, partial order, scattered, coloring, rationals

References

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