Reverse mathematics and well-scattered partial orders

Emanuele Frittaion

Dipartimento di Matematica e Informatica, University of Udine, viale delle Scienze 206, 33100 Udine, Italy. emanuele.frittaion@uniud.it, http://sole.dimi.uniud.it/~emanuele.frittaion/

Abstract. I will discuss *well-scattered partial orders* in the context of reverse mathematics.

We say that P is a well-scattered partial order (WSPO) if for every function $f: \mathbb{Q} \to P$ there exist $x <_{\mathbb{Q}} y$ such that $f(x) \leq_{P} f(y)$. The following theorem gives four classically equivalent definitions for wspo's.

Theorem 1 (Bonnet, Pouzet 1969). Let P be a partial order. Then the following are equivalent:

- 1. P is a wspo;
- 2. P is scattered and has no infinite antichains;
- 3. every linear extension of P is scattered;
- 4. for every function $f: \mathbb{Q} \to P$ there exists an infinite set $A \subseteq \mathbb{Q}$ such that $x <_{\mathbb{Q}} y$ implies $f(x) \leq_P f(y)$ for all $x, y \in A$.

We want to compare the above definitions from the viewpoint of reverse mathematics. For example, what's necessary (and sufficient) to prove the statement "every linear extension of a wspo is scattered"?

It turns out that, except for the statements already provable in RCA_0 , there are roughly two families: statements provable in WKL_0 but not in RCA_0 (neither in $WWKL_0$) and statements provable in ACA_0 but not in WKL_0 . With regard to the second family, the following partition theorem comes into play.

Theorem 2 (Erdös, Rado 1952). The partition relation $\mathbb{Q} \to (\aleph_0, \mathbb{Q})^2$ holds, that is for every coloring c: $[\mathbb{Q}]^2 \to 2$ there exists either an infinite 0-homogeneous set or a dense 1-homogeneous set.

Let ER_2^2 (after Erdös-Rado) be the formal statement corresponding to Erdös-Rado theorem. The reverse mathematics of ER_2^2 is interesting by itself as it lies between ACA_0 and RT_2^2 . I still do not know whether it is strictly between them. As for the relation with wspo's, I will discuss semitransitive versions of ER_2^2 .

Keywords: Reverse Mathematics, partial order, scattered, coloring, rationals

References

- 1. PETER A. CHOLAK, ALBERTO MARCONE AND REED SOLOMON, Reverse mathematics and the equivalence of definitions for well and better quasi-orders, J. Symbolic Logic, 69(3), pages 683–712, 2004.
- 2. EMANUELE FRITTAION, Reverse Mathematics and partial orders, PhD thesis, 2014.