Proof Theory of Quantified Modal Logics

Indexed modal logics (IMLs), see [1], constitute the object of study of this paper. IMLs generalize quantified modal logics (QMLs) in two respects: language and semantics. First of all, standard modal operators $\Box$ and $\Diamond$ are replaced by modal operators indexed by sets of pairs composed by a term and a variable: $|^{t_1}_{x_1} \dots {}^{t_n}_{x_n}|$ and $\langle^{t_1}_{x_1} \dots {}^{t_m}_{x_m}\rangle$. This allows us to distinguish between 'it is necessary for c to be P' and 'it is necessary that c is P', which are expressed by $|^{c}_{x}|P(x)$ and $\Box P(c)$, respectively. In this approach we can better control the interaction of first-order machinery (substitutions, quantifiers, and identity) with modalities. The second novelty is that Kripke semantics is replaced by the more general transition semantics, in which the relation of trans-world identity, used to evaluate modal open formulas, is replaced by an arbitrary relation between objects inhabiting possible worlds. This allows us to have a finer-grained correspondence theory than that of Kripke semantics: many important formulas that are valid on every Kripke-frame correspond to particular classes of transition-frames.

IMLs are a major step in the model-theoretic understanding of quantified modal logics. Their proof theory has been confined to axiomatic systems, see [1], for which completeness results are very hard to find and in most cases are still open problems. Our approach is different because we replace axiomatic systems with labelled sequent calculi. These calculi allow us to internalize transition semantics into the rules of inferences of the calculus, and to make use of the method of axioms-as-rules, which has already been used in [2] for propositional modal logics. In this way we are able to define sequent calculi for many interesting semantically defined classes of transition-frames.

We prove the following general results for our calculi: the structural rules of weakening and contraction are height-preserving admissible, and the rule of cut is admissible. Then we prove, in a modular way, that each calculus is sound and complete with respect to the corresponding class of transition-frames. We stress that these results are mostly new, e.g. in [1] it was possible to give a completeness result for the minimal IML (with and without rigid designators), but not for any of its extensions.

Our works is, to our knowledge, the first attempt to merge two of the more active fields of research in modal logics: that of generalizations of Kripke semantics and that of proof-theoretical studies of modal logics. We believe that our results show clearly the advantages of our approach.

References

[1] Corsi, G. (2009). 'Necessary for'. In *Log., Method. and Phil. Sci., 13th Congress*, pp. 162–184. London: King's College Publications.

[2] Negri, S. (2005). 'Proof Analysis in Modal Logic'. *Journal of Philosophical logic*, 34: 507–544.