

Closed subspaces in pointfree Topology

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We connect two seemingly distant concepts in pointfree Topology: *positivity relations* in Formal Topology and *lower powerlocales* in the theory of Locales.

Positivity relations [4] were invented to deal with the notion of closure in Formal Topology (= intuitionistic, predicative, pointfree Topology). The same idea can be transferred to locales. Every positivity relation on a locale corresponds to a family (actually a complete lattice) of sublocales, each of which is *overt and weakly closed* [5] (which just means *closed* if classical logic is assumed) and hence it is a point of the lower powerlocale [1].

We are thus led to a neat description of the category of locales with positivity relations (known as *positive topologies*, in the predicative case). An object is a locale together with a suplattice of distinguished points of its lower powerlocale. An arrow is a morphism between locales which induces a mapping between the distinguished points.

This characterization makes evident that (i) every locale can be equipped with a canonical positivity relation, constructed as in [2], and that (ii) under such an identification the category of locales becomes a reflective subcategory of the category of (large) positive topologies (= locales with positivity relations).

A natural question is whether the points selected by a positivity relation comes from a sublocale of the lower powerlocale. For instance, the canonical positivity corresponds to the whole lower powerlocale. A *localic suplattice* [3] is an algebra for the lower powerlocale monad (so lower powerlocales are free localic suplattices). The points of a localic sub-suplattice of the lower powerlocale give rise to a positivity relation on the underlying locale. With classical logic, the converse holds as well.

References

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