Injectivity of relational semantics with respect to MELL proof-nets and the Taylor expansion

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Starting from investigations on denotational semantics of System F (second order typed λ-calculus), in 1987 Girard [Gir87] introduced linear logic (LL), a refinement of intuitionistic logic. He defines two new modalities, ! and ?, giving a logical status to structural rules and allowing to distinguish between linear resources (i.e. usable exactly once during the cut-elimination process) and resources available at will. One of the main features of LL is the possibility of representing proofs geometrically (so as the λ-calculus terms) by means of particular graphs, the proof-structures. Among proof-structures it is possible to characterize "in a geometric way" the ones corresponding to proofs in LL sequent calculus through the Danos-Regnier correctness criterion ([DR89]): a proof-structure is a proof-net (i.e. it corrisponds to a proof in LL sequent calculus) if and only if it fulfills some conditions about acyclicity and connectedness (ACC).

Ehrhard [Ehr05] introduced finiteness spaces, a denotational model of LL (and λ-calculus) which interprets formulas by topological vector spaces and proofs by analytical functions: in this model the operations of differentiation and the Taylor expansion make sense. Ehrhard and Regnier [ER03, ER06, ER08] internalized these operations in the syntax and thus introduced differential linear logic DiLL (and differential λ-calculus), where the promotion rule (the only one in LL which is responsible for introducing the !-modality and hence creating resources available at will) is replaced by three "finitary" rules which are perfectly symmetric to the rules for the ?-modality: this allows a more subtle analysis of the resources consumption during the cut-elimination process. At the syntactic level, the Taylor expansion decomposes a LL proof-structure in a (generally infinite) formal sum of DiLL proof-structures, each of which contains resources usable only a fixed number of times.

Our contribution aims at looking further into the relationship between the Taylor expansion and the relational model. The relational model is one of the well-known and simplest denotational semantics of LL and λ-calculus: it interprets LL proof-structures as morphisms in the category of sets and relations. More precisely:

1. We show that, given a cut-free and η-expansed MELL (the multiplicative-exponential fragment of LL, sufficiently expressive to encode the λ-calculus) proof-structure π, each point of the Taylor expansion of π is identical to one and only one element of the set of injective points of the interpretation of π in the relational model, quotiented by the equivalence relation induced by atoms renaming. This does not hold if π contains cuts, consistently with the idea that the Taylor expansion of a MELL proof-structure can be seen as an object between syntax and semantics (which is invariant under cut-elimination).

2. We show that every MELL proof-structure fulfilling the ACC condition is uniquely determined by the pair of points of order 1 and 2 in its Taylor expansion (in order to do a comparison with mathematical analysis, this means that the analytical functions fulfilling some condition are uniquely determined by their first and second derivatives). In order to obtain this result, we adapt to the new framework of DiLL some well-known tools of the theory of LL proof-nets (in particular the notion of empire, see [Gir87]).

3. As a corollary of points 1 and 2, we show that the relational model is injective with respect to the MELL proof-structures fulfilling ACC (i.e. given two cut-free and η-expansed MELL proof-nets, if they have the same relational interpretation then they are identical). The injectivity of the relational model in the more general case of MELL proof-structures without weakenings has already recently been proved by de Carvalho and Tortora de Falco in [dCT12]. Our proof, which represents a remarkable simplification, follows a completely different approach based on the notion of Taylor

expansion. All these works fit in the general perspective of abolishing the old traditional distinction between syntax and semantics.

We think that our result mentioned in point 2 can be improved: we conjecture that every MELL proof-net is uniquely determined by the point of order 2 of its Taylor expansion. This study also pushes towards a deeper understanding of the Taylor expansion of a MELL proof-structure, which should lead to a more abstract and synthetic representation of it.

References

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