An extention to the notion of MV-algebras: fMV-algebras

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MV-algebras are the algebraic counterpart of Łukasiewicz ∞-valued logic. An MV-algebra is a structure $(A, \oplus, ^*, 0)$, where$(A, \oplus, 0)$ is an abelian monoid, * is an involution and the condition $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$ is satisfied for all $x, y \in A$.

A fundamental result in the theory of MV-algebras is their categorical equivalence with the category of abelian lattice-ordered groups with strong unit, proved by Mundici in [5]. Moreover, MV-algebras can be endowed with products: an internal product has been introduced in [2] and an external product in [3], leading to the notion of Product MV-algebra and Riesz MV-algebra respectively. This structures turn out to be categorical equivalent to lattice-ordered rings with strong unit and Riesz Spaces.

The aim of our work is to follow this path, and consider structures endowed with both internal and external product. They are connected, by extensions of Mundici's categorical equivalence, with f-algebras.

Definition 1 An fMV-algebra A is an algebraic structure $(A, \star, \cdot, \oplus, {}^*, 0)$ where $\cdot$ and $\oplus$ are binary operations, * is unary, 0 is a constant and $\star : [0, 1] \times A \to A$ is such that:
(F1) $(A, \cdot, \oplus, {}^*, 0)$ is a PMV-algebra,
(F2) $(A, \star, \oplus, {}^*, 0)$ is a Riesz MV-algebra,
(F3) $r \star (x \cdot y) = (r \star x) \cdot y = x \cdot (r \star y)$ for any $r \in [0, 1]$ and $x, y \in A$,
(F4) $(z \cdot (x \odot y^*)) \wedge (y \odot x^*) = 0$ for any $x, y, z \in A$.

Recall that an f-algebra V is an f-ring endowed with a structure of Riesz space such that $\alpha(x \cdot y) = (\alpha x) \cdot y = x \cdot (\alpha y)$ for any $\alpha \in \mathbb{R}$ and $x, y \in V$. They were introduced by Birkhoff and Pierce in [1]. A *unital* f-algebra is an f-algebra with strong unit such that the strong unit is also unit for the product. The first step in a deep study of fMV-algebra is the following theorem.

Theorem 2 *The category of fMV-algebras is equivalent with the category of f-algebras with strong unit with unit-preserving morphisms.*

Some special classes of fMV-algebras arises from general theory of f-algebras: we say that a fMV-algebra is *formally real* if it belongs to HSP([0, 1]) and we denote by $\mathbb{FR}$ the class of formally real fMV-algebras. By well-known results of universal algebra, the free fMV-algebra in $\mathbb{FR}$ exists and its elements are term functions defined on $[0, 1]$. It follows that

$FR_n = \{\widetilde{t} \mid t \in Term_n,\ \widetilde{t} : [0,1]^n \to [0,1] \text{ is the term function of } t\}$.
Moreover

Proposition 3 *The elements of FR_n are continuous piecewise polynomial functions defined on the n-cube.*

The converse of the above proposition is related to the Birkhoff-Pierce conjecture [1]. Since the conjecture is true for $n \leq 2$ [4], we get the following.

Theorem 4 *For $n \leq 2$, the f-MV-algebra FR_n is the set of all continuous piecewise polynomial functions defined on the n-cube, i.e any continuous piecewise polynomial function defined on the n-cube is a term function from FR_n.*

Another important class is the one of *semiprime* fMV-algebras. We will call a fMV-algebra semiprime if it satisfies the quasi identity $(x^2 = 0 \Rightarrow x = 0)$, for any element x in the algebra. For unital, commutative and semiprime fMV-algebras, is it possible to prove, between other results, the standard subdirect representation theorem with respect to chains and the extension of Di Nola's embedding in non standard reals.

We connect semiprime and formally real fMV-algebra by the following theorem.

Theorem 5 *For an fMV-algebra A the following are equivalent:*
(1) $A \in$ ISP([0,1]),
(2) A is a unital, commutative and semiprime fMV-algebra,
(3) A is in $\mathbb{FR}$ and $x \cdot x = 0$ implies $x = 0$ for any $x \in A$.

Finally, it is possible to define the logic of such structure that in the semiprime case is complete with respect to $[0,1]$.

References

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