Non standard truths

Toward a model-theoretic taxonomy of (conservative) axiomatic theories of truth

In a standard approach, an axiomatic theory of truth is formulated by adding new axioms (or rules) governing a truth predicate to a base theory. The base theory is supposed to give necessary information on the objects to which the truth predicate is intended to apply, namely to prove basic facts about syntax. At the same time, it usually plays the role of the object theory, namely the theory whose truth wants to be characterized. This dual role can be achieved by taking an arithmetical theory (in a suitable setting) as base/object theory and proceeding via arithmetization of syntax. For such reasons, PA (Peano Arithmetic), in one of its usual first order formulations, is the most common choice. PA has the further virtue of being a very well known theory both under a proof-theoretic and a model-theoretic perspective.

At this point the language in which PA is carried out, call it '$\mathcal{L}_{PA}$', is expanded with a new unary predicate 'Tr', and PA is extended with axioms governing the behaviour of this new symbol in such a way that it can reasonably be taken to act as a truth predicate. Clearly, a range of different choices are possible here, so that we gain a number of different theories of truth. Starting from simple systems based on Tarskian biconditionals -namely sentences of the form $(Tr(\ulcorner\varphi\urcorner) \leftrightarrow \varphi)$, where '$\ulcorner\varphi\urcorner$' is the gödelian for the $\mathcal{L}_{PA}$-sentence 'φ'- to more refined systems yielded from compositional clauses and allowing for iterations of the truth predicate.

These different axiomatic theories of truth are usually compared in terms of their proof theoretic strength. Usual techniques give us relevant answers in terms of relative interpretations, proof-theoretic reductions or *conservative extensions*. Model theoretic means are often used in this study, but they are frequently relegated at a secondary role, as mere tools to obtain proof theoretic results.

I propose to reverse this attitude and offer a different taxonomy by considering the models of PA that can be expanded to models of the various truth theories. In particular, I will focus on some conservative (over PA) theories of truth. Conservative theories are unable to prove new theorems in $\mathcal{L}_{PA}$ (with respect to those already proven by PA), so that, from a proof theoretic point of view they seem arithmetically alike. Nevertheless, from a model theoretic perspective, they can be arranged along a scale obtaining a finer-grained classification. Indeed, simple conservative theories of truth can be shown to fit different models of PA in many cases. This strategy gives us both logical information and new grounds for philosophical reflections.