Dualities and geomtry

Luca Spada[a,b]

[a]*ILLC, Universiteit van Amsterdam, Science Park 107, 1098XG Amsterdam, The Netherlands.*
[b]*Dipartimento di Matematica. Università degli Studi di Salerno. Via Giovanni Paolo II 132, 84084 Fisciano (SA), Italy.*

In classical algebraic geometry, one studies the solutions sets over an algebraically closed field k of systems equations between polynomials in n variables with coefficients in k. An *affine set* is any subset of affine space k^n. An affine set is an *affine variety* if it is the solution set $V \subseteq k^n$ of some system of equations. Any affine variety V comes with its own *coordinate ring*, i.e., the quotient of the polynomial ring $k[x_1, \ldots, x_n]$ by the ideal generated by any system of equations that defines V. In fact, there is a (contravariant) Galois connection between subsets of k^n and of $k[x_1, \ldots, x_n]$ which carries a subset $E \subseteq k[x_1, \ldots, x_n]$ to the affine set $\mathbb{V}(E) \subseteq k^n$ of common zeros of the polynomials in E; and carries an affine set $S \subseteq k^n$ to the set (indeed, ideal) $\mathbb{I}(S)$ of polynomials in $k[x_1, \ldots, x_n]$ vanishing over S. The affine sets of k^n that are fixed by this Galois connection are, trivially, the affine varieties. The ideals of $k[x_1, \ldots, x_n]$ that are fixed by the Galois connection are, non-trivially, precisely the *radical ideals*, i.e. those ideals that coincide with the intersection of all prime ideals containing them. The latter statement is the content of Hilbert's *Nullstellensatz*; in symbols, $\mathbb{I}(\mathbb{V}(I)) = I$ if, and only if, I is radical. Stated otherwise, I is radical if, and only if, $k[x_1, \ldots, x_n]/I$ is the coordinate ring of an affine variety.

The aim of this talk is to show that all of the above generalises to any equationally definable class of algebras V (*variety of algebras* in the sense of universal algebra). For this, replace the ground field k by an arbitrary V-algebra A. Replace the polynomial ring $k[X]$ over the (possibly infinite) set of variables X by the free V-algebra Free_X generated by X. Replace affine (possibly infinite-dimensional) space k^X by the direct power A^X. Replace ideals of $k[X]$ by congruences on Free_X. Replace affine sets by A-affine sets, i.e. subsets of A^X. Replace polynomial maps by term-definable maps. The Galois pair $(\mathbb{V}, \mathbb{I})$ then admits a direct generalisation to this context, and the Galois connections lifts to a dual adjunction between the category of all V-algebras, and the category of A-affine sets and term-definable maps. An abstract form of Hilbert's *Nulllstellensatz* holds in this general context, and characterises the congruences θ on Free_X such that $\mathbb{I}(\mathbb{V}(\theta)) = \theta$, that is, the *$A$-radical congruences*; the V-algebras Free_X/θ for such θ's are the *coordinate V-algebras* of the A-affine set $\mathbb{V}(\theta)$.

The induced duality between the full subcategories on the fixed objects provides a link between the *Nullstellensatz*, and topological dualities in the style of Marshall Stone for varieties of algebras. I will discuss how several dualities of this sort, such as Stone, Priestley, Gelfand and Pontryagin duality, flow naturally from these general results.

I will conclude by expounding a further generalisation of the basic adjunction between systems of equations and their sets of solutions at the level of arbitrary categories that satisfy a minimal set of assumptions. This general framework affords an account of all the foregoing plus e.g., Galois theory of field extensions, or the adjoint connection between subgroups of the fundamental group of a path-connected and semi-locally simply connected topological space and its path-connected covering spaces.

Based on joint work in progress with Olivia Caramello (IHES), and Vincenzo Marra (Milan).