Interpreting Łukasiewicz logic into Intuitionistic logic.

Vincenzo Marra, Daniel McNeill,[1] Andrea Pedrini

Key words: Łukasiewicz logic, Intuitionistic logic, MV-algebra, Heyting algebra, Consequence relation, Polyhedral geometry.

Łukasiewicz (infinite-valued propositional) logic – denoted L – and Intuitionistic (propositional) logic – denoted Int – are two of the oldest and most well studied systems of non-classical logic. See [2] and [1], respectively, for background. The two logics were conceived with entirely different motivations, and have very different formal properties. Nonetheless, we prove the following.

Notation. We fix countable sets of propositional variables X and Y, and write $\textsc{Form}_X$ for the set of formulæ of L and $\textsc{Form}_Y$ for the set of formulæ of Int. We further denote by $\vdash_{\mathsf{L}}$ and $\vdash_{\mathsf{Int}}$ the syntactic consequence relations of L and Int, respectively.

Theorem. *There exists a deductively closed theory Θ_{L} in Int, and a function $\mathscr{T}\colon \textsc{Form}_X \to \textsc{Form}_Y$, such that, for each $\alpha \in \textsc{Form}_X$, the following holds.*

$$\vdash_{\mathsf{L}} \alpha \qquad \textit{if, and only if,} \qquad \Theta_{\mathsf{L}} \vdash_{\mathsf{Int}} \mathscr{T}(\alpha) \tag{1}$$

Remark. The function $\mathscr{T}$ is essentially surjective in the following sense. Given $\beta_1 \in \textsc{Form}_Y$, there exists $\beta_2 \in \textsc{Form}_Y$ such that β_1 and β_2 are logically equivalent modulo Θ_{L} (i.e. $\Theta_{\mathsf{L}} \vdash_{\mathsf{Int}} \beta_1 \leftrightarrow \beta_2$), and $\mathscr{T}(\alpha) = \beta_2$ for some $\alpha \in \textsc{Form}_X$.

The proof of the theorem rests on a remarkable property of the lattice $\mathscr{L}_{\mathsf{fa}}$ of finitely axiomatisable theories in L:

Lemma. *The (distributive) lattice $\mathscr{L}_{\mathsf{fa}}$ is a (countable)Heyting algebra.*

The set of maximally consistent theories in L carries a natural topology that makes it homeomorphic to $[0,1]^\omega$. The lattice $\mathscr{L}_{\mathsf{fa}}$ is anti-isomorphic to the lattice of (cylindrified) rational polyhedra in $[0,1]^\omega$. This is proved by passing to Lindenbaum-Tarski algebras, and applying the geometric duality theory of Chang's MV-algebras, the algebraic counterparts of L. Algebraically, the lemma asserts the remarkable fact that *the lattice of principal ideals of $\mathscr{F}_{\mathsf{L}}$, the free MV-algebra on ω generators, is a countable Heyting algebra.* This result is part of a more general investigation of the topology of prime spectral spaces of MV-algebras and related structures; see Andrea Pedrini's submitted abstract. It follows that there is an onto homomorphism of Heyting algebras

$$q\colon \mathscr{F}_{\mathsf{Int}} \twoheadrightarrow \mathscr{L}_{\mathsf{fa}},$$

where $\mathscr{F}_{\mathsf{Int}}$ is the free Heyting algebra on ω generators. Now, if $\top$ is the top element of $\mathscr{L}_{\mathsf{fa}}$, the filter $q^{-1}(\top)$ corresponds to a theory Θ_{L} in Int, and the map q can be used to define the translation map $\mathscr{T}$, leading to a proof of the theorem.

At the time of writing, the theorem above is a purely existential result. In further work, we plan to investigate the properties of $\mathscr{T}$ and Θ_{L} more closely. Some obvious questions to be addressed include axiomatisability of Θ_{L}, and computability of $\mathscr{T}$.

[1]Presenting author. Dipartimento di Scienze Teoriche e Applicate, Università degli Studi dell'Insubria, Via Mazzini, 5, 21100 Varese, Italy. E-mail danmcne@gmail.com

Abstract submitted to AILA2014 on Tuesday 4^{th} March, 2014

References

[1] A. Chagrov and M. Zakharyaschev. *Modal logic*, volume 35 of *Oxford Logic Guides*. The Clarendon Press Oxford University Press, New York, 1997. Oxford Science Publications.

[2] R. L. O. Cignoli, I. M. L. D'Ottaviano, and D. Mundici. *Algebraic foundations of many-valued reasoning*, volume 7 of *Trends in Logic—Studia Logica Library*. Kluwer Academic Publishers, Dordrecht, 2000.