The space of directions of a polyhedron.

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We study the Stone-Priestley dual space of the lattice of subpolyhedra of a compact polyhedron, with motivations coming from geometry, topology, ordered-algebra, and non-classical (especially Łukasiewicz) logic. Due to space constraints, in this abstract we assume sone familiarity with Stone-Priestley duality and polyhedral geometry.

Recall that a *polytope* in $\mathbb{R}^n$ is the convex hull of a finite subset of $\mathbb{R}^n$. Polytopes are thus compact and convex. A *polyhedron* in $\mathbb{R}^n$ is any subset that can be written as the union of finitely many polytopes. Polyhedra are thus compact, but not necessarily convex. Given a polyhedron $P \subseteq \mathbb{R}^n$, let $\operatorname{Sub} P$ denote the collection of all polyhedra contained in P. Observe that $\operatorname{Sub} P$ is a distributive lattice under intersections and unions, with top element P and bottom element $\emptyset$.

Let $\operatorname{Spec} \operatorname{Sub} P$ be the spectral space of prime filters of $\operatorname{Sub} P$, equipped with the dual Stone topology. The main result we announce here is that $\operatorname{Spec} \operatorname{Sub} P$ has a concrete description in terms of a non-Hausdorff completion of the space P which holds great geometric interest. The lattice $\operatorname{Sub} P$ is an instance of a *Wallman basis* of the topological space P. This leads to the natural (Wallman) embedding $P \hookrightarrow \operatorname{Spec} \operatorname{Sub} P$ that extends P from a space consisting of points to one consisting of *directions*. Informally, a first-order direction in a polyhedron P is a point $p \in P$ together with the germ of a half line springing from p, an initial segment of which is contained in P. Higher order directions replace segments with simplices. We now give a precise statement of our main result.

We denote by $\operatorname{conv}\{w_1, \ldots, w_k\}$ the convex hull of the points $w_1, \ldots, w_k \in \mathbb{R}^n$, by $\mathscr{H}(w_1, \ldots, w_k)$ the hyperplane of $\mathbb{R}^n$ orthogonal to each w_i, and by $\mathbb{S}^{n-1}$ the unit $(n-1)$-sphere in $\mathbb{R}^n$.

Definition. The *space of directions* of P is the set $\Delta(P) := \bigcup_{k=0}^{n} \Delta_k(P)$, with each layer $\Delta_k(P)$ inductively defined as

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\begin{aligned}
\Delta_0(P) &:= \{p \in \mathbb{R}^n \mid p \in P\} = P, \\
\Delta_1(P) &:= \{(p, v_1) \mid p \in \Delta_0(P),\ v_1 \in \mathbb{S}^{n-1} \text{ and } \exists \varepsilon_1 > 0 \text{ s.t. } \operatorname{conv}\{p, p + \varepsilon_1 v_1\} \subseteq P\} \\
\Delta_k(P) &:= \{(p, v_1, \ldots, v_k) \mid (p, v_1, \ldots, v_{k-1}) \in \Delta_{k-1},\ v_k \in \mathbb{S}^{n-1} \cap \mathscr{H}(v_1, \ldots, v_{k-1}) \text{ and} \\
&\qquad \exists \varepsilon_1, \ldots, \varepsilon_k > 0 \text{ such that } \operatorname{conv}\{p, p + \varepsilon_1 v_1, \ldots, p + \varepsilon_1 v_1 + \cdots + \varepsilon_k v_k\} \subseteq P\}.
\end{aligned}
$$

The topology of $\Delta(P)$ is generated by the basis of closed sets $\{\Delta(Q) \mid Q \in \operatorname{Sub} P\}$.

There is a map

$$\mathrm{I}\colon\ \Delta(P) \longrightarrow \operatorname{Spec} \operatorname{Sub} P$$

that takes a direction $\delta \in \Delta(P)$ to the collection of subpolyhedra Q of P containing it, by which we mean that $\delta \in \Delta(Q)$. Main result:

Theorem. *The map* I *is a homeomorphism.*

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Remark. In [1], Panti classified by geometric means the prime ℓ-ideals of free finitely generated vector lattices and lattice-ordered Abelian groups, using the notion of direction above. His main tool is the use of directional derivatives of piecewise-linear functions. While we cannot offer a full comparison of the two results here, we direct the reader's attention to the following key points. (1) Our result is independent of the theory of vector lattices and lattice-groups. (2) We do not use piecewise linear maps, nor their derivatives. Everything is encoded by filters of closed polyhedral sets. (3) We remove the algebraic restriction of freeness, which geometrically corresponds to assuming that P is homeomorphic to a sphere. (4) Our motivations are different; our result is a part of long-term project of understanding the PL topology of polyhedra in terms of their non-Hausdorff completion $\Delta(P)$.

We prove this result by direct geometric arguments of some length. If time allows, we discuss selected consequences of the main result, including compactness of the subspace of minimal primes of $\operatorname{Spec}\operatorname{Sub} P$, and the following fundamental property of $\operatorname{Sub} P$.

Lemma. *The lattice* $\operatorname{Sub} P$ *is a co-Heyting algebra. Equivalently, its order-dual* $\operatorname{Sub} P^{\mathrm{op}}$ *— isomorphic to the lattice of open subpolyhedra of* P *— is a Heyting algebra.*

By extending the proof of the lemma above somewhat, and applying the geometric duality theory of Chang's MV-algebras, we are able to obtain the following:

Corollary. *The lattice of principal MV-ideals of the free MV-algebra on* ω *generators is a Heyting algebra.*

Please see Dan McNeill's submitted abstract for the logical import of the preceding corollary in the context of Łukasiewicz logic.

References

[1] Giovanni Panti. Prime ideals in free l-groups and free vector lattices. *J. Algebra*, 219(1):173–200, 1999.