

FROM WQOS TO NOETHERIAN SPACES: SOME REVERSE MATHEMATICS RESULTS

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I'll present work in progress which involves also Emanuele Frittaion, Matthew Hendtlass, Paul Shafer, and Jeroen Van der Meeren.

If (Q, \leq_Q) is a quasi-order we can equip Q with several topologies. We are interested in the Alexandroff topology (the closed sets are exactly the downward closed subsets of Q) and the upper topology (the downward closures of finite subsets of Q are a basis for the closed sets). We denote these topologies by $A(Q)$ and $u(Q)$ respectively. Except in trivial situations, $A(Q)$ and $u(Q)$ are not T_1 , yet they reflect several features of the quasi-order. For example, (Q, \leq_Q) is a well quasi-order (WQO: well-founded and with no infinite antichains) if and only if $A(Q)$ is Noetherian (all open sets are compact or, equivalently, there is no strictly descending chain of closed sets). Moreover, if (Q, \leq_Q) is WQO then $u(Q)$ is Noetherian. Noetherian spaces originate in algebraic geometry, because the Zariski topology of a Noetherian ring is Noetherian.

Given the quasi-order (Q, \leq_Q) , there are two natural quasi-orders on the powerset $\mathcal{P}(Q)$:

$$\begin{aligned} A \leq^b B &\iff \forall a \in A \exists b \in B a \leq_Q b; \\ A \leq^\sharp B &\iff \forall b \in B \exists a \in A a \leq_Q b. \end{aligned}$$

We write $\mathcal{P}^b(Q)$ and $\mathcal{P}^\sharp(Q)$ for the resulting quasi-orders, and $\mathcal{P}_f^b(Q)$ and $\mathcal{P}_f^\sharp(Q)$ for their restrictions to the collection of finite subsets of Q .

Goubault-Larrecq ([GL07]) proved that if (Q, \leq_Q) is WQO then $u(\mathcal{P}^b(Q))$ and $u(\mathcal{P}_f^\sharp(Q))$ are Noetherian, even though $\mathcal{P}^b(Q)$ and $\mathcal{P}_f^\sharp(Q)$ are not always WQOs. These results were then applied to infinite-state verification problems ([GL10]).

We study these theorems and some of their consequences from the viewpoint of reverse mathematics, proving for example:

- over RCA_0 , ACA_0 is equivalent to each of “if (Q, \leq_Q) is WQO then $u(\mathcal{P}^b(Q))$ is Noetherian”, and “if (Q, \leq_Q) is WQO then $A(\mathcal{P}_f^b(Q))$ is Noetherian”;
- ACA_0 proves “if (Q, \leq_Q) is WQO then $u(\mathcal{P}_f^\sharp(Q))$ is Noetherian”, yet WKL_0 does not.

REFERENCES

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