Very Large Cardinals and Combinatorics

Vincenzo Dimonte

February 27, 2014

Large cardinals are currently one of the main areas of investigation in Set Theory. They are possible new axioms for mathematics, and they have been proven essential in the analysis of the relative consistency of mathematical propositions. It is particularly convenient the fact that these hypotheses are neatly well-ordered by consistency strength, therefore giving a meaningful tool of comparison between different hypotheses. It is natural to ask how flexible can be the set-theoretical universe under large cardinals assumptions. In other words, once a large cardinal hypotheses is assumed, which structural characteristics are admissible in the universe? Which combinatorial principles are consistent or inconsistent?

Specific importance have $V = L$, that is in contradiction with all the stronger large cardinals, and various combinatoric properties entailed by it. Examples of these properties are: GCH (or, more weakly, SCH), principles like $\diamondsuit$ and $\square$ and variations of them, and so on. It is of course of interest, to better gauge the range of possible universes under large cardinal assumptions, to investigate also the exact opposite of such principles, and to describe how far from L a universe, combinatorics-wise, can be.

My contribution will delineate the state-of-the-art of this research applied to the large cardinal hypotheses that are at the top of the large cardinal hierarchy: rank-into-rank embeddings. They are some of the strongest hypotheses and postulate the existence of embeddings among ranks of the Levy hierarchy. They imply (at least consistency-wise) all the commonly used large cardinals, and they have a peculiar character. I3, for example, was seminal for braid group theory, while I0 has great (yet not fully explained) similarities with the Axiom of Determinacy.

A deep analysis of Easton forcing and Prikry forcing lead to the desired results: all the very large cardinals are consistent with L-like properties like GCH, $\diamondsuit$, $V = \mathrm{HOD}$, while under I0 we have also the consistency of I1 and the opposite of those principles, like the negation of SCH and others.