Choice sequences in proof theory

(Summary)

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Proof systems are collections of production rules by which formal derivations are defined inductively. Properties of derivations are usually proved by first giving transformations that bring derivations to some standard form, followed by a proof that derivations in standard form have the desired property. A prime example is the normalization procedure in natural deduction and the subformula property of normal derivations.

An arbitrary derivation can be considered a given initial object d_0 to which any one of a number of transformations applies, with some derivation d_1 as a result. Repeated transformations give a perfect example of a *choice sequence* α, an initial segment $\langle \alpha_1 \dots \alpha_n \rangle$ of which produces the sequence of derivations $d_1 = \alpha_1(d_0)$, $d_2 = \alpha_2(d_1), \dots, d_n = \alpha_n(d_{n-1})$. Well known methods from the theory of choice sequences can therefore be applied to establish properties of proof systems. The initial object and its repeated transformations generate a tree, and properties are typically established by proofs of well-foundedness. The following properties, all defined by quantification over choice sequences, are given as examples of such proofs:

1. *Normalizability:* A derivation d in natural deduction is defined to be normalizable if there exists a choice sequence α of conversion steps such that an initial segment $\langle \alpha_1 \dots \alpha_n \rangle$ brings d to normal form. Normalizability follows by a double induction in which it is shown that logical rules maintain normalizability.

2. *Strong normalizability:* A derivation d in natural deduction is defined to be strongly normalizable if for all choice sequences α of conversion steps, there is an n such that $\langle \alpha_1 \dots \alpha_n \rangle$ brings d to normal form. The proof of strong normalizability follows by the fan theorem variety of bar induction.

3. *Reducibility in arithmetic:* The notion of a *reduction sequence*, definition omitted here, applies to sequents $\Gamma \rightarrow C$ in a formal system of arithmetic. In what is historically the first use of choice sequences in proof theory, Gentzen proved in 1935 the termination of the reduction sequences. As a third example of choice sequences in proof theory, it is shown how the termination proof can be made formal though an explicit use of bar induction.