A Kleene realizability model for the Minimalist Foundation

Maria Emilia Maietti[1], Samuele Maschio[1] and Takako Nemoto[2]

[1] Dipartimento di Matematica, Università di Padova, Italia
[2] JAIST, Nomi, Japan.

Abstract

A foundation for mathematics should be called constructive only if the mathematics arising from it could be considered genuinely computable. One way to show this is to produce a realizability model of the foundation where arbitrary sets are interpreted as data types and functions between them are interpreted as programs. A key example is Kleene's realizability model for first-order Intuitionist Arithmetics.

In this talk we will show how to build a realizability model for the Minimalist Foundation, ideated by Maietti and Sambin in [MS05] and then completed by Maietti in [Mai09], where it is explicit how to extract programs from its proofs.

One novelty of the Minimalist Foundation is that it consists of two levels: an extensional level formulated in a language close as much as possible to that of ordinary mathematics, and an intensional level formulated in a language suitable for computer-aided formalization of proofs and program extraction.

Since the extensional level is interpreted in the intensional one in [Mai09] by making use of a quotient completion (see [MR13]), to build a realizability model for the whole foundation it is enough to build it for its intensional level. So we build a model for the intensional level by extending original Kleene's realizability in the theory $\widehat{ID_1}$ and using Beeson's techniques in [Bee85]. The theory $\widehat{ID_1}$ is formulated in the language of second-order arithmetics and it consists of PA (Peano Arithmetic) plus the existence of some (not necessary the least) fixed point for positive parameter-free arithmetical operators.

An important consequence of building a Kleene realizability semantics for the intensional level of the Minimalist Foundation is to get a proof that such a level satisfies the proofs-as-programs requirement advocated in [MS05], namely it is consistent with the axiom of choice (AC) and the formal Church thesis (CT).

This result represents an important and necessary breakthrough to understand in what sense the Minimalist Foundation provides an abstract notion of computable functions between all its entities.

References

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