

A Kleene realizability model for the Minimalist Foundation

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Abstract

A foundation for mathematics should be called constructive only if the mathematics arising from it could be considered genuinely computable. One way to show this is to produce a realizability model of the foundation where arbitrary sets are interpreted as data types and functions between them are interpreted as programs. A key example is Kleene's realizability model for first-order Intuitionist Arithmetics validating the formal Church thesis.

In this talk we will show how to build a realizability model validating the formal Church thesis for the Minimalist Foundation, ideated by Maietti and Sambin in [MS05] and then completed by Maietti in [Mai09], where it is explicit how to extract programs from its proofs.

One novelty of the Minimalist Foundation is that it consists of two levels: an extensional level formulated in a language close as much as possible to that of ordinary mathematics, and an intensional level formulated in a language suitable for computer-aided formalization of proofs and program extraction.

Since the extensional level is interpreted in the intensional one in [Mai09] by making use of a quotient completion (see [MR13]), to build a realizability model for the whole foundation it is enough to build it for its intensional level. So we build a model for the intensional level by extending original Kleene's realizability in the theory \widehat{ID}_1 and using Beeson's techniques in [Bee85]. The theory \widehat{ID}_1 is formulated in the language of second-order arithmetics and it consists of PA (Peano Arithmetic) plus the existence of some (not necessary the least) fixed point for positive parameter-free arithmetical operators.

This result represents an important and necessary breakthrough to understand in what sense the Minimalist Foundation provides an abstract notion of computable functions between all its entities.

References

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