Homotopy theory in definably complete structures

Antongiulio Fornasiero

Let M be a definably complete (DC) expansion of an ordered field. When M is o-minimal, several authors (Berarducci, Otero, Peterzil, Starchenko, etc.) have shown how to transfer various tools from differential topology to definable sets and definable maps. One important such tool is the degree of a continuous definable map from the circumference to itself: similarly to the classical case, it is a well-defined integer number, which is invariant under definable homotopies. From this fact, Brouwer's fixed point theorem for definable continuous maps follows easily. We show that when M is not o-minimal, but only DC, the degree of a map is no longer an integer, but an element of a suitable "ring of cardinalities", defined in a way similar to the Grothendieck ring of M of Krajicek and Scanlon. Again, the degree of a map well-defined and invariant under definable homotopies. Under a suitable pigeon-hole assumption, such ring is nontrivial, and hence, as usual, Brouwer's fixed point theorem for definable maps follows.